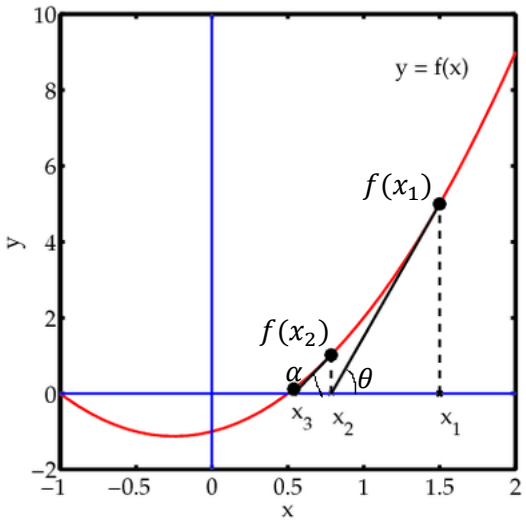
Newton Raphson Method

The Newton-Raphson method is a technique for approximating the roots of a continuous and differentiable function that uses the tangent line to the function at each iteration. The method aims to quickly find a good approximation for the root of the function by approximating it with a straight line that is tangent to the function. This method is also known as **Tangent Method.** Newton Raphson method falls under **Non-Bracketing** method.

The Newton-Raphson method is an iterative method, and its convergence can be very fast if the function f(x) is well-behaved and the initial guess x_1 is sufficiently close to the true root. However, the method may fail to converge or converge to a wrong root if the function has a complicated behavior or if the initial guess is far from the true root. Therefore, it is important to carefully choose the initial guess and to check the convergence of the method.

Derivation of the N-R Formula



Consider a graph of a function, f(x) shown in the above figure. Let, x_1 be the approximate root of f(x) = 0. Now, draw a tangent at $x = x_1$ and the point of intersection of this tangent and x-axis gives the second approximation of the root.

From the figure, for the initial tangent with an angle θ, the slope is given by:

$$\tan\theta = \frac{f(x_1)}{x_1 - x_2}$$

i.e. $f'(x_1) = \frac{f(x_1)}{x_1 - x_2}$

$$x_1 - x_2 = \frac{f(x_1)}{f'(x_1)}$$

$$\therefore x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

Similarly,

$\tan\alpha = \frac{f(x_2)}{x_2 - x_3}$

i.e. $f'(x_2) = \frac{f(x_2)}{x_2 - x_3}$

$x_2 - x_3 = \frac{f(x_2)}{f'(x_2)}$

$\therefore x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$

So, in general, $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

Algorithm

Step 1: Start with an initial guess for the root of the equation, denoted by x_1 and error e.

Step 2: Evaluate $f(x_1)$ and $f'(x_1)$

Step 3: Calculate the next estimate of the root using the formula:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

Step 4: If $\left|\frac{x_2 - x_1}{x_2}\right|$ > e,

Set $x_1 = x_2$ and repeat 3-4

Step 5: Display x_2 as root

Step 6: Stop

Newton Raphson Method for Mathematical deductions

1. Using Newton Raphson Method, show that $\frac{1}{N}\ is\ x_{n+1} = x_n(2 - Nx_n)$

   Solution:

   Let, $x = \frac{1}{N}$

$$\frac{1}{x} - N = 0$$

So, $f(x) = \frac{1}{x} - N$

$f'(x) = -x^{-2}$

We know, the N-R Formula is given by:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$
$$= x_n - \frac{\left(\frac{1}{x} - N\right)}{(-x_n^{-2})}$$
$$= x_n + \left(\frac{1}{x} - N\right) x_n^2$$
$$= x_n + x_n - N x_n^2$$
$$= 2\, x_n - N x_n^2$$
$$= x_n(2 - N x_n)$$
$$\therefore x_{n+1} = x_n(2 - N x_n)$$

2. Using Newton Raphson Method, show that for $\sqrt{N}$, $x_{n+1} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right)$.

Solution:

Let, $x = \sqrt{N}$

$x^2 - N = 0$

So, $f(x) = x^2 - N$

$f'(x) = 2x$

We know, the N-R Formula is given by:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$
$$= x_n - \frac{x_n^2 - N}{2x_n}$$
$$= \frac{2x_n^2 - x_n^2 + N}{2x_n}$$
$$= \frac{x_n^2 + N}{2x_n}$$
$$= \frac{1}{2}\left(x_n + \frac{N}{x_n}\right)$$
$$\therefore x_{n+1} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right)$$

Examples

1. Find the root of $3x = \cos(x) + 1$ using Newton Raphson Method, correct upto 3 decimal places.

   Solution:

   Given:

   $3x = \cos(x) + 1$

   Let, $f(x) = 3x - \cos(x) - 1$

   Then,

   $f'(x) = 3 + \sin(x)$

   We know, the N-R Formula is given by:

   $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

   $$x_{n+1} = x_n - \frac{3x_n - \cos(x_n) - 1}{3 + \sin(x_n)}$$

   Let, the initial guess, $x_1 = 0$

   $x_2 = x_1 - \frac{3x_1 - \cos(x_1) - 1}{3 + \sin(x_1)} = 0.6667$

   $x_3 = x_2 - \frac{3x_2 - \cos(x_2) - 1}{3 + \sin(x_2)} = 0.6074$

   $x_4 = x_3 - \frac{3x_3 - \cos(x_3) - 1}{3 + \sin(x_3)} = 0.6071$

   Hence, the root is 0.6071, which is correct upto 3 decimal places.

2. Find the root of the equation $3x + \sin(x) - e^x = 0$ using Newton Raphson Method correct upto 4 decimal places.

   Solution:

   Given:

   $3x + \sin(x) - e^x = 0$

   Let, $f(x) = 3x + \sin(x) - e^x$

   Then,

   $f'(x) = 3 + \cos(x) - e^x$

   We know, the N-R Formula is given by:

   $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

   $$x_{n+1} = x_n - \frac{3x_n + \sin(x_n) - e^{x_n}}{3 + \cos(x_n) - e^{x_n}}$$

   Let, the initial guess, $x_1 = 0$

   $$x_2 = x_1 - \frac{3x_1 + \sin(x_1) - e^{x_1}}{3 + \cos(x_1) - e^{x_1}} = 0.33333$$

$$x_3 = x_2 - \frac{3x_2 + \sin(x_2) - e^{x_2}}{3 + \cos(x_2) - e^{x_2}} = 0.36017$$

$$x_4 = x_3 - \frac{3x_3 + \sin(x_3) - e^{x_3}}{3 + \cos(x_3) - e^{x_3}} = 0.36042$$

$$x_5 = x_4 - \frac{3x_4 + \sin(x_4) - e^{x_4}}{3 + \cos(x_4) - e^{x_4}} = 0.36042$$

Hence, the root is 0.36042 which is correct upto 4 decimal places.

Practice Questions:

1. Find the root of the equation $e^{2x} - 3x = 0$ using Newton Raphson Method correct upto 4 decimal places.
2. Find the root of the equation $x^3 + x^2 - 3x - 3 = 0$using Newton Raphson Method correct upto 4 decimal places.
3. Evaluate $\sqrt{28}$ using Newton Raphson Method correct upto 4 decimal places.
4. Find the root of the equation $\log(x) - \cos(x) = 0$ using Newton Raphson Method correct upto 3 decimal places.
5. Find the root of the equation $3x - e^x = 0$ using Newton Raphson Method correct upto 4 decimal places.
6. Using Newton Raphson Method, show that for $\frac{1}{\sqrt{N}}$, $x_{n+1} = \frac{1}{2}\left(x_n + \frac{1}{Nx_n}\right)$