Fixed-Point Iteration Method

The fixed-point iteration method is a numerical technique used to approximate the roots of a function. It is based on the concept of a fixed point, which is a point at which a function f(x) takes on the same value as x.

The basic idea of the fixed-point iteration method is to start with an initial guess x_0, and then use the function f(x) to generate a sequence of values x1, x2, x3, and so on, by repeatedly applying the function to the previous value.

Consider a function $f(x) = 0$ .....(i) which can be transformed to the form $x = g(x)$ .....(ii) such that x is on the left side of the equation. This transformation can be done by simplifying an algebraic expression or by adding x on both sides of the given equation $f(x) = 0$.

Thus, if x_0 is the initial guess, the next approximation is given by:

$x_1 = g(x_0)$

The iteration form can be generalized as:

$x_{n+1} = g(x_n)$

Which is the required formula for Fixed-Point Iteration Method.

Note: The form of x = g(x) can be chosen in many ways. But we choose g(x) for which |g'(x)|<1 at x = x_o.

Algorithm

Step 1: Start

Step 2: Initialize the value of x_0 and e

Step 3: Calculate $x_1 = g(x_0)$

Step 4: If $\left|\frac{x_1 - x_0}{x_1}\right|$ < e, go to step 5

Otherwise, set $x_0 = x_1$ and go to step 3

Step 5: Display x_1 as root

Step 6: Stop

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Divergence in Fixed-Point Iteration Method

The fixed-point iteration method may not always converge to a root of the function f(x). If the transformation function g(x) is poorly chosen or the absolute value of the derivative of g(x) at the fixed point is greater than or equal to 1, the iteration may diverge and not converge to any root. This is called divergence in the fixed-point iteration method. The following are the types of divergence we may encounter while following this method:

1. Oscillatory Divergence

   Oscillatory divergence occurs when the sequence of iterates oscillates back and forth around the fixed point, without ever converging to it. This can happen when the absolute value of the derivative of the function f(x) is greater than 1 in a neighborhood of the fixed point. In this case, the magnitude of the error in each iteration increases, leading to the oscillation of the sequence around the fixed point.

   Consider a function, $f(x) = x^2 - 5 = 0$
   So, $x = \frac{5}{x}$
   i.e. $g(x) = \frac{5}{x}$

   Let, the initial guess, x_0 = 1
   So, $x_1 = g(x_0) = \frac{5}{1} = 5$
   $x_2 = g(x_2) = \frac{5}{5} = 1$
   $x_3 = g(x_2) = \frac{5}{1} = 5$
   $x_4 = g(x_3) = \frac{5}{5} = 1$

   Thus, the above form is known as Oscillatory Divergence.

2. Monotonous Divergence

   Monotonous divergence, on the other hand, occurs when the sequence of iterates diverges away from the fixed point in a monotonous manner, without oscillating. This can happen when the absolute value of the derivative of the function f(x) is exactly 1 at the fixed point. In this case, the magnitude of the error in each iteration remains constant, leading to a steady increase in the distance between the iterates and the fixed point.

   Consider a function, $f(x) = x^2 - 5 = 0$
   So, $x^2 = 5$
   $x^2 + x = 5 + x$ (adding x on both sides)
   $x = x^2 + x - 5$
   i.e. $g(x) = x^2 + x - 5$

   Let, the initial guess, x_0 = 0
   So, $x_1 = g(x_0) = 0^2 + 0 - 5 = -5$

$x_2 = g(x_1) = (-5)^2 - 5 - 5 = 15$
$x_3 = g(x_2) = 15^2 + 15 - 5 = 235$
$x_4 = g(x_3) = 235^2 + 235 - 5 = 55455$

Thus, the above form is known as Monotonous Divergence.

Examples

1. Find the root of the equation $x^2 - 3x + 2 = 0$ correct up to 3 decimal places using Fixed-Point Iteration Method.

   Solution:

   Given:

   $f(x) = x^2 - 3x + 2 = 0$

   So, $x^2 + 2 = 3x$

   $\therefore x = \dfrac{x^2 + 2}{3} = g(x)$

   Initially, let $x_0 = 0$

   $x_1 = \dfrac{{x_0}^2 + 2}{3} = 0.6667$

   $x_2 = \dfrac{{x_1}^2 + 2}{3} = 0.8148$

   $x_3 = \dfrac{{x_2}^2 + 2}{3} = 0.8879$

   $x_4 = \dfrac{{x_3}^2 + 2}{3} = 0.9294$

   $x_5 = \dfrac{{x_4}^2 + 2}{3} = 0.9545$

   $x_6 = \dfrac{{x_5}^2 + 2}{3} = 0.9734$

   $x_7 = \dfrac{{x_6}^2 + 2}{3} = 0.9825$

   $x_8 = \dfrac{{x_7}^2 + 2}{3} = 0.9884$

   $x_9 = \dfrac{{x_8}^2 + 2}{3} = 0.9923$

   $x_{10} = \dfrac{{x_9}^2 + 2}{3} = 0.9948$

   $x_{11} = \dfrac{{x_{10}}^2 + 2}{3} = 0.9965$

   $x_{12} = \dfrac{{x_{11}}^2 + 2}{3} = 0.9976$

   $x_{13} = \dfrac{{x_{12}}^2 + 2}{3} = 0.9984$

$$x_{14} = \frac{{x_{13}}^2 + 2}{3} = 0.9989$$

Hence, 0.9989 is the root which is correct upto 3 decimal places.

2. Find the root of the equation $3x + \sin(x) - e^x = 0$ using Fixed Point Method correct upto 4 decimal places.

Solution:

Given:

Let, $f(x) = 3x + \sin(x) - e^x = 0$

$3x + \sin(x) - e^x = 0$

$3x = e^x - \sin(x)$

$\therefore x = \frac{e^x - \sin(x)}{3} = g(x)$

Initially, let x_0 = 1

$$x_1 = \frac{e^{x_0} - \sin(x_0)}{3} = 0.62560$$
$$x_2 = \frac{e^{x_1} - \sin(x_1)}{3} = 0.42792$$
$$x_3 = \frac{e^{x_2} - \sin(x_2)}{3} = 0.36256$$
$$x_4 = \frac{e^{x_3} - \sin(x_3)}{3} = 0.36047$$
$$x_5 = \frac{e^{x_4} - \sin(x_4)}{3} = 0.36042$$

Hence, 0.36042 is the root which is correct up to 4 decimal places.