Unit 1: Solution of Nonlinear Equations

Introduction

Nonlinear equations are mathematical expressions that describe a relationship between two or more variables where the relationship is not proportional or linear. In other words, nonlinear equations are those that involve at least one term that has a power greater than one, or products or ratios of variables, which create curves and nonlinear relationships between the variables.

For any problem of the form **f(x) = 0,** the solution process includes finding the value of **x** that will satisfy the equation **f(x) = 0.**

The solution to a nonlinear equation is a set of values for the variables that satisfies the equation. In other words, the solution is the point or points on the graph of the equation where it intersects with the x- or y-axis, or where the function evaluates to zero. Finding the solution to nonlinear equations can be a challenging task, and it may require analytical or numerical methods to obtain an accurate solution.

For example, consider the equation y = x^2 - 4. The solution to this equation is the value(s) of x that satisfy the equation. To find the solution analytically, we can set y = 0 and solve for x:

0 = x^2 - 4

x^2 = 4

x = ±2

Therefore, the solutions to the equation y = x^2 - 4 are x = 2 and x = -2.

In some cases, it may not be possible to find an exact analytical solution, and numerical methods may be required to obtain an approximation of the solution. Iterative methods such as the bisection method, the Newton-Raphson method, and the secant method, or direct methods such as the Gauss elimination method and LU decomposition, can be used to obtain an approximate solution to a nonlinear equation. The choice of method depends on the nature of the equation and the desired accuracy of the solution.

Characteristics of Numerical Methods

Numerical methods are techniques used to approximate solutions to mathematical problems that cannot be solved analytically. Some characteristics of numerical methods include:

1. Approximation: Numerical methods are designed to provide approximate solutions to mathematical problems, rather than exact solutions. The accuracy of the approximation depends on the chosen algorithm and the precision of the input data.

2. Iterative: Most numerical methods are iterative in nature, meaning that they involve a repeated sequence of calculations that approach the desired solution. The number of iterations required

to achieve an acceptable level of accuracy depends on the complexity of the problem and the chosen algorithm.

3. Algorithmic: Numerical methods rely on well-defined algorithms that prescribe a sequence of steps to solve a problem. The effectiveness of a numerical method depends on the quality of the algorithm and its implementation.

4. Numerical stability: Numerical methods can be numerically stable or unstable. A stable method produces accurate results even when small changes are made to the input data or algorithm, while an unstable method can produce wildly different results for small changes in input data or algorithm.

5. Convergence: Numerical methods must converge to a solution that is accurate enough for the problem at hand. The rate of convergence depends on the problem and the algorithm, and it can be affected by the choice of initial values, step sizes, and other parameters.

6. Computational complexity: Numerical methods can have different levels of computational complexity, which affects the time and resources required to solve a problem. The choice of method depends on the problem size, accuracy requirements, and available computational resources.

Types of Equation

Equations can be broadly classified into two categories: linear equations and nonlinear equations. Linear equations are those that can be expressed in the form ax + b = 0, where a and b are constants and x is the variable. Nonlinear equations are those that cannot be expressed in this form. Based on this classification, we can further categorize equations into the following types:

1. Linear equations: As mentioned above, linear equations are those that can be expressed in the form ax + b = 0. Examples of linear equations include 3x + 5 = 11, 2x - 4 = 10, and 4x + 2y = 8. Linear equations have a straight line as their graph and can be solved easily using algebraic methods.

2. Nonlinear equations: Nonlinear equations are those that cannot be expressed in the form ax + b = 0. They can be further classified into several types based on their degree and the type of nonlinear function involved. Examples of nonlinear equations include x^2 + y^2 = 1, sin(x) + cos(y) = 1, and ln(x) + 2x = 3. Nonlinear equations can have different types of graphs, such as parabolic, exponential, logarithmic, trigonometric, or other types of curves, and they can be solved using various analytical or numerical methods.

Some common types of nonlinear equations include:

a) Quadratic equations: Quadratic equations are nonlinear equations of degree two and have the form $ax^2 + bx + c = 0$. Examples include $x^2 + 3x - 4 = 0$ and $2x^2 - 5x + 3 = 0$. Quadratic equations have a parabolic curve as their graph.

b) Cubic equations: Cubic equations are nonlinear equations of degree three and have the form $ax^3 + bx^2 + cx + d = 0$. Examples include $x^3 + 2x^2 - 5x + 6 = 0$ and $3x^3 - 7x + 2 = 0$. Cubic equations have a curve that can be concave up or concave down as their graph.

c) Exponential equations: Exponential equations are nonlinear equations that involve an exponential function, such as $y = a^x$ or $y = e^x$. Examples include $2^x - 4 = 0$ and $e^x - 5x + 2 = 0$. Exponential equations have a curved shape and can be increasing or decreasing, depending on the sign of the exponent.

d) Trigonometric equations: Trigonometric equations are nonlinear equations that involve trigonometric functions, such as sine, cosine, and tangent. Examples include $\sin(x) + \cos(x) = 1$ and $2\tan(x) - 3 = 0$. Trigonometric equations have different types of graphs, such as sine waves and cosine waves.

Prepared By: Pawan Niroula| pawanniroula777@gmail.com

Errors in Computing

Error is the **difference** or the **deviation** from **exact** or the **true** values. In numerical methods, errors can arise from various sources, such as approximations, round-off errors, truncation errors, and convergence errors.

Types of Errors

1. Inherent Error
   Inherent Errors are the errors that are already present in the data supplied to the model/equations. Inherent error can be considered as input error. Inherent error can be further divided into two types:
   a. Data Error
      They are the type of inherent errors which arise when data supplied for a problem are obtained by some experimental means and hence are of limited accuracy and precision. This may be due to some limitations in instrumentation and reading which may be unavoidable. It is also known as **Empirical Error.**

   b. Conversion Error
      Conversion error, also known as **Representational Error** arise due to the limitation of the computer to store the data exactly. For instance, the floating point number 0.1010203050222222…… is represented with specified number of decimal points only.
2. Numerical Error
   Numerical error, also known as **Procedural Error** are introduced in the process or iteration of implementation of numerical methods. Numerical error can be further divided into:

a) Truncation error: This error arises from approximating a mathematical function using a finite number of terms. The approximation may not represent the true value of the function accurately, and the error between the true value and the approximation is called the truncation error. This type of error can be reduced by increasing the number of terms used in the approximation.

   Suppose we want to approximate the value of π using the formula $\pi/4 = 1 - 1/3 + 1/5 - 1/7 + \ldots$. If we stop after the first three terms, the approximation is $1 - 1/3 + 1/5 = 0.93333$. The truncation error is the difference between this approximation and the true value of $\pi/4$.

b) Round-off error: This error arises from rounding off decimal places in intermediate calculations. Computers use a finite number of digits to represent numbers, so any calculation that involves more digits than the computer can handle will be rounded off. This can lead to small errors in the final result, known as round-off error. This error can be reduced by using higher-precision arithmetic or by carefully designing algorithms to minimize the number of intermediate calculations.

   If we calculate the value of 1/7 using a calculator that only displays two decimal places, we get the approximation 0.14. However, the true value of 1/7 is 0.142857..., which cannot be

represented exactly in two decimal places. The round-off error is the difference between the true value and the approximation.

Errors based on Quantity

1. Absolute Error
   Absolute error is a measure of how far "off" a measurement is from a **true value** or an indication of the uncertainty in a measurement.
   For instance, if you measure the width of a book using a ruler with millimeter marks, the best you can do is measure the width of the book to the nearest millimeter. You measure the book and find it to be **80mm**. After measuring, you report the absolute error in the measurement as **80mm$\pm$1 mm.** The **absolute error** is **1mm.**

   Absolute error is calculated as : $|X_t - X_a|$ where, X_t is the **true value** and X_a is the **calculated value.**

2. Relative Error
   Relative error expresses how large the **absolute error** is compared with the total size of the object you are concerned with. It tells us whether the error of the measured value is big or small.
   Relative error is calculated as: $\left|\frac{X_t - X_a}{X_t}\right|$

3. Percentage Error
   Percent error is the difference between estimated value and the actual value in comparison to the actual value and is expressed as a percentage. In other words, the percent error is the relative error multiplied by 100.
   Percentage error is calculated as: $\left|\frac{X_t - X_a}{X_t}\right|$ * 100%

Methods for solution of non-linear equations

Bracketing and open end methods are two approaches for finding the roots of a function, which is the point(s) at which the function equals zero.

1. Bracketing method: This method involves identifying an interval in which the root is known to exist and then narrowing down that interval until the root is found with the desired accuracy. This method is also known as the "interval halving" method or the "bisection" method.

   The common bracketing method includes the following:

   a. Bisection Method
   b. False-Position Method

2. Non-Bracketing Method: This method is also known as **Open-End Method.** This method is used to find the roots of a function without requiring prior knowledge of an interval in which the root

exists. These methods typically require only a single initial guess, and then iteratively refine that guess until the root is found.

The common non-bracketing method includes the following:

a. Newton Raphson Method
b. Secant Method
c. Fixed-Point Method

Non-bracketing methods can converge faster than bracketing methods when the initial guess is close to the root, but they may fail to converge or converge to a different root if the initial guess is too far away. Therefore, it is important to choose a good initial guess and to carefully monitor the convergence of the solution.

The Bisection Method

Bisection Method is the simplest and easiest **bracketing** approach to find the solution of non-linear equation. It is a simple, robust and reliable algorithm that involves repeatedly bisecting an interval and selecting the subinterval in which the root lies until a desired level of accuracy is achieved. The bisection method is based on **Intermediate Value Theorem.**

Intermediate Value Theorem states that:

"If a function *f(x)* is continuous in the interval [x_1, x_2] and *f(x_1)* and *f(x_2)* have opposite signs, then the equation *f(x) = 0* has at least one root between $x=x_1$ and $x=x_2$."

Here, the midpoint of the curve, between x_1 and x_2 is calculated as:

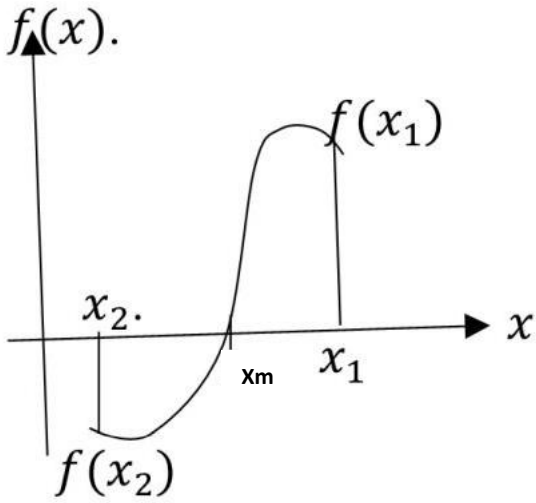
$x_m = \frac{x_1+x_2}{2}$

When we calculate the above mid-point, we obtain two new intervals :

1. [x_1, x_m]
2. [x_m, x_2]

To calculate the roots, we have the following conditions:

1. If $f(x_m) = 0$ then *root* = x_m.
2. If $f(x_1)*f(x_m) < 0$, *root* lies between x_1 and x_m.
3. If $f(x_m)*f(x_2) < 0$, *root* lies between x_m and x_2.



Algorithm

Step 1: Start

Step 2: Initialize the values of x_1, x_2, e

Step 3: Compute $f_1 = f(x_1)$ and $f_2 = f(x_2)$

Step 4: If $f_1 * f_2 > 0$, then display **wrong initial guess as roots are not within x_1 and x_2** and goto (2)

Else

Continue

Step 5: Calculate the mid-point as : $x_m = \frac{x_1+x_2}{2}$

Step 6: If $f(x_1)*f(x_m) < 0$ then $x_2 = x_m$

Else $x_1 = x_m$

Step 7: If $|f(x_m)| > e$ then goto 5 else goto 8

Step 8: Display x_m as root

Step 9: Stop

Convergence of Bisection Method

In bisection method, interval is halved in each iteration. After n^{th} iteration, the size of interval is reduced to:

$$\frac{x_2-x_1}{2^n} = \frac{\Delta x}{2^n}$$

From the above, we can conclude that the maximum error after n^{th} iteration is: $E_n = \pm\frac{\Delta x}{2^n}$

So, $|E_n| = \frac{\Delta x}{2^n}$

If we extend this to $(n+1)^{th}$ iteration, the error is given by :

$$|E_{n+1}| = \frac{\Delta x}{2^{n+1}} = \frac{\Delta x}{2^n * 2} = \frac{E_n}{2}$$

The above equation indicates that the error is halved after each iteration of the bisection method. Thus, we can derive a conclusion that **bisection method converges linearly.**

Examples

1. Find a root of an equation $f(x) = x^3 + x^2 - 3x - 3$ using Bisection Method, correct up to 3 decimal places.

   Solution:

   Given:

   $f(x) = x^3 + x^2 - 3x - 3$

   Let, the initial guess i.e. $x_1 = 1$ and $x_2 = 2$.

   Since, Bisection method is the bracketing method, we need to check if the initial guess brackets the root or not.

   Now, we calculate $f(x_1)$ and $f(x_2)$.

   $f(x_1) = 1^3 + 1^2 - 3 * 1 - 3$

   $= 1+1-3-3$

   $= -4$

   $f(x_2) = 2^3 + 2^2 - 3 * 2 - 3$

   $= 8+4-6-3$

   $= 3$

   Since, $f(x_1) * f(x_2) < 0$, root lies between $x_1 = 1$ and $x_2 = 2$.

Iteration	x_1	x_2	$x_m = \frac{x_1+x_2}{2}$	$f(x_1)$	$f(x_2)$	$f(x_m)$	Update
1	1	2	1.5	-4	3	-1.8	$x_1 = x_m$
2	1.5	2	1.75	-1.8	3	0.1	$x_2 = x_m$
3	1.5	1.75	1.625	-1.8	0.1	-0.9	$x_1 = x_m$
4	1.625	1.75	1.6875	-0.9	0.1	-0.4	$x_1 = x_m$
5	1.6875	1.75	1.7187	-0.4	0.1	-0.12	$x_1 = x_m$
6	1.7187	1.75	1.7343	-0.12	0.1	0.02	$x_2 = x_m$
7	1.7187	1.7343	1.7265	-0.12	0.02	-0.05	$x_1 = x_m$
8	1.7265	1.7343	1.7304	-0.05	0.02	-0.01	$x_1 = x_m$
9	1.7304	1.7343	1.7325	-0.01	0.02	0.002	$x_2 = x_m$
10	1.7304	1.7325	1.7314	-0.01	0.004	-0.005	$x_1 = x_m$
11	1.7314	1.7325	1.7319	-0.06	0.004	0.0009	$x_2 = x_m$

Hence, the root is **1.7319** which is correct upto 3 decimal places.

2. Find the root of x = cos(x) using Bisection Method correct upto three decimal places.

Solution:

Given:

x = cos(x)

Let, *f(x)* = x-cos(x)

Let, the initial guess i.e. x_1 = 0 and x_2=1.

Since, Bisection method is the bracketing method, we need to check if the initial guess brackets the root or not.

Now, we calculate $f(x_1)$ and $f(x_2)$.

$f(x_1)$ = 0-cos(0)

= 0 - 1

= -1

$f(x_2)$ = 1 – cos(1)

= 1 – 0.540

= 0.45

Since, $f(x_1) * f(x_2) < 0$, root lies between x_1 = 0 and x_2=1.

Iteration	x_1	x_2	$x_m = \frac{x_1+x_2}{2}$	$f(x_1)$	$f(x_2)$	$f(x_m)$	Update
1	0	1	0.5	-1	0.45	-0.37	$x_1 = x_m$
2	0.5	1	0.75	-0.37	0.45	0.01	$x_2 = x_m$
3	0.5	0.75	0.625	-0.37	0.01	-0.18	$x_1 = x_m$
4	0.625	0.75	0.6875	-0.18	0.01	-0.08	$x_1 = x_m$
5	0.6875	0.75	0.7187	-0.08	0.01	-0.03	$x_1 = x_m$
6	0.7187	0.75	0.7343	-0.03	0.01	-0.007	$x_1 = x_m$
7	0.7343	0.75	0.7421	-0.007	0.01	0.005	$x_2 = x_m$
8	0.7343	0.7421	0.7382	-0.007	0.005	-0.001	$x_1 = x_m$
9	0.7382	0.7421	0.7401	-0.001	0.005	0.001	$x_2 = x_m$
10	0.7382	0.7401	0.7391	-0.001	0.001	0.0001	$x_2 = x_m$
11	0.7382	0.7391	0.7386	-0.001	0.00002	-0.0007	$x_1 = x_m$
12	0.7386	0.7391	0.7388	-0.0008	0.00002	-0.0003	$x_1 = x_m$

Practice Questions:

1. Find the root of $e^{-x} - x = 0$ correct upto 3 decimal places using bisection method.
2. Find the root of $sinx = \frac{1}{x}$ correct upto 2 decimal places using bisection method.
3. Find the root of $x - \cos(x) = 0$ correct upto 4 decimal places using bisection method.
4. Find the root of $3x + \sin(x) - e^x = 0$ correct upto 4 decimal places using bisection method.
5. Find the root of $\ln(x) - x + 3 = 0$ correct upto 4 decimal places.